BRST Cohomology and String Field Theory

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Chapter 1

Introduction

We shall be interested in the quantization of irreducible gauge systems with,for simplicity, bosonic degrees of freedom only.By an irreducible gauge system we mean that all the constraints are linearly independent. Generally speaking there are two methods of approach: 1) the “reduced phase space method” and 2) the Becchi-Rouet-Stora-Tyupin (BRST) method. We begin our discussion in the frame of Hamiltonian mechanics (leading in the quantum level to what is called a first quantized theory), although later we shall focus on field theory (classical and quantum). In particular, this piece of work wishes to formulate the classical BRST field theory method in the geometric context of jet bundles.

So we begin with a system with phase space P and let $C^\infty(P)$ be the algebra of smooth phase space functions. In addition, we require a symplectic structure to be defined on P, so $C^\infty(P)$ becomes a Poisson algebra.

In order to quantize the system, we must find the space of the classical observables at first.

We consider only irreducible systems. This is done by a two-step reduction procedure: (i) We identify the functions that coincide on the constraint surface Σ which means that we replace the algebra $C^\infty(P)$ by the algebra $C^\infty(\Sigma)$. (ii) We impose the gauge invariance condition on the elements of $C^\infty(\Sigma)$ in order to obtain the space of (classical) observables. In the simplest case when one has only first class primary constraints the gauge condition means that an observable is a phase space function that has vanishing Poisson bracket with the constraints. In a more geometric language, this means that they must be annihilated by the vertical exterior derivative d on Σ (to be defined in the sequel).

Once we have defined the space of (classical) observables, we can canonically quantize the system using the correspondence:

$$(\text{Poisson bracket}) \to (i\hbar)^{-1}(\text{commutator})$$

This prescription is a consistent one in that the objects involved have the same algebraic properties (eg. Jacobi identity).

At this point, before we embark on describing the BRST method in general, we think we must make some comments on ghosts. The reason for this is that ghosts are really important in quantum theory and essential for the BRST formalism. However if the method is clearly understood in the classical level, then passing to the quantum level is straightforward.

Ghosts were first encountered in Quantum Field Theory (QFT) as fields with the "wrong" relation between spin and statistics. They enabled one to maintain locality and they also ensured that the theory would be unitary and independent of gauge choice. They are not "real" particles in the sense that they do all the above tasks by contributions to virtual processes only.

After the advent of the Fadeev-Popov(F-P) measure for summing over equivalence classes of gauge field histories, the ghost were regarded as just an artifact leading to a useful representation of that measure. They were restored to a somewhat more respectable role when it was realised that in more complicated theories it was necessary to include self-interactions of the ghosts in order to achieve unitarity and this could not be fitted in the F-P scheme.

However, it was the discovery of BRST symmetry that raised the ghosts to a prominent role. Since this symmetry mixes the ghosts with other fields, it became clear that one should regard all the fields, including the ghosts , as different components of a single geometrical object. This point of view emerged as the logical development of the idea of gauge invariance.

The need for ghosts and the symmetry that reveals their profound importance were first established in the quantum level. It was only afterwards realised that they have a natural and necessary place within classical mechanics as well, although we must emphasise that ghosts should not have direct physical meaning within classical mechanics.

The main idea of BRST formalism is to substitute for the original gauge symmetry a fermionic rigid symmetry, (rigid means independent of space-time and fermionic means that it involves fermionic degrees of freedom ,i.e. anti-commuting variables), acting on an appropriately extended phase space. The important point is that symmetry completely captures the original gauge invariance and leads to a simpler formlualtion of the theory.

The advantages of the BRST formalism are the following: 1) The replacement of a gauge symmetry by a rigid symmetry enables one to substitute for the original gauge invariant action one that depends on all the variables and that can be used directly in the path integral. By contrast, the original action cannot be used as such in the path integral, since the integration over the gauge degrees of freedom yields infinity since the integrand does not depend on these variables. This infinity is compensated for by the ghosts. Compensations between the ghosts and original degrees of freedom may also happen in other instances (eg. avoidance of anomalies). 2) The use of BRST symmetry allows one to work consistently with functions defined in the original phase space, thus preserving manifest locality and covariance.

Chapter 2

Mathematical Interlude

In this chapter we shall present in brief some of the mathematical tools needed for a rigorous treatment of the BRST formalism.

2.1 Jet Bundles

Let (E, π, M) be a bundle and let $p\epsilon M$. Define the local sections $\phi, \psi, \epsilon\Gamma_p(\pi)$ to be "1-equivalent" at p if $\phi(p) = \psi(p)$ and if in some adapted coordinate system (x^i, u^a) around $\phi(p)$

$$\frac{\partial\phi^a}{\partial x^i} \mid_p = \frac{\partial\psi^a}{\partial x^i} \mid_p$$

for $1 \leq i \leq m$ and $1 \leq a \leq n$. The equivalence class containing ϕ is called "1-jet of ϕ at p" and is denoted $j^1_p\phi$ The "first jet manifold" of π is the set

$$J^1\pi := \{j^1_p\phi : p\epsilon M, \phi\epsilon\Gamma_p(\pi)\}$$

If (x^i, u^a) is an adapted coordinate system on E, then the induced coordiante system on $J^1\pi$ is (x^i, u^a, u^a_i) where $x^i(j^1_p\phi) = x^i(p)$, $u^a(j^1_p\phi) = u^a(\phi(p))$ and $u^a_1(j^1_p\phi) = \frac{\partial\phi^a}{\partial x^i} \mid_p$ ¿From the definiton of $J^1\pi$, two natural bundles arise: (i) $(J^1\pi, \pi_1, M)$, where $\pi_1 : J^1\pi \to M$, $j^1_p\phi \mapsto p$ and (ii) $(J^1\pi, \pi_{1,0}, E)$, where $\pi_{1,0} = J^1\pi \to E$, $j^1_p\phi \mapsto \phi(p)$ The second is an affine bundle modelled on the vector bundle

$$(\tau{*}_E \mid_{\pi*(T*M)}) \otimes (\tau_E \mid_{V\pi})$$

where of course τ denotes the tangent bundle. If (E, π, M) is a bundle, $W \subset M$ open submanifold and $\phi\epsilon\Gamma_W(\pi)$, then the "first prolongation of ϕ" is the section $j^1\phi\epsilon\Gamma_W(\pi_1)$ defined by

$$j^1\phi(p) = j^1_p\phi, \qquad p\epsilon W$$

Analogously one can defined "second jet manifolds" : Let (E, π, M) be a bundle and let $p\epsilon M$. Define the local sections $\phi, \psi\epsilon\Gamma_p(\pi)$ to be "2-equivalent" at

p if $\phi(p) = \psi(p)$ and if in some adapted coordinate system (x^i, u^a) around $\phi(p)$

$$\frac{\partial \phi^a}{\partial x^i} |_p = \frac{\partial \psi^a}{\partial x^i} |_p \text{ and } \frac{\partial^2 \phi^a}{\partial x^i \partial x^j} |_p = \frac{\partial^2 \psi^a}{\partial x^i \partial x^j} |_p$$

for $1 \leq i, j \leq m$ and $1 \leq a \leq n$. The equivalence class containing ϕ is called "2 jet of ϕ at p" and it is denoted $j^2_p \phi$ The "second jet manifold" of π is then the set

$$J^2 \pi := \{ j^2_p \phi : p \epsilon M, \phi \epsilon \Gamma_p(\pi) \}$$

The induced coordinate system on $J^2\pi$ is then $(x^i, u^a, u^a_i, u^a_{ij})$ where $x^i(j^2_p\phi) = x^i(p),\ u^a(j^2_p\phi) = u^a(\phi(p)), u^a_i(j^2_p\phi) = u^a_i(j^1_p\phi)$ and $u^a_{ij}(j^2_p\phi) = \frac{\partial^2 \phi^a}{\partial x^j \partial x^i} |_p$

¿From the above definition,three natural bundles arise:
(i) $(J^2\pi, \pi_2, M),\ \pi_2 : J^2\pi \rightarrow M,\ j^2_p\phi \mapsto p$
(ii) $(J^2\pi, \pi_{2,0}, E),\ \pi_{2,0} : J^2\pi \rightarrow E,\ j^2_p\phi \mapsto \phi(p)$
(iii) $(J^2\pi, \pi_{2,1}, J^1\pi),\ \pi_{2,1} : J^2\pi \rightarrow J^1\pi,\ j^2_p\phi \mapsto j^1_p\phi$ The third one is an affine bundle modelled on the vector bundle

$$(S^2 \tau *_{j^1\pi} |_{\pi *_1 (S^2 T * M)}) \otimes \pi *_{1,0} (\tau_E |_{V\pi})$$

where S denotes the symmetric product. Similarly, the "second prolongation" of a section ϕ is the section $j^2\phi\epsilon\Gamma_W(\pi_2)$ defined by

$$j^2\phi(p) = j^2_p\phi,\ p\epsilon W$$

The notion of a jet manifold can be further extended to include also "∞-order jet manifolds". They arise as inverse limits of certain functors and they are ∞-dimensional Fréchet manifolds (but not Banach manifolds). For more details about jet bundles see [1].

2.2 Homological Perturbation Theory

Let A be an algebra. A "homological resolution" of A is an N-graded differential algebra $\bar{A}$ with differential δ of degree minus one such that

$$H_k(\delta) = 0,\ k \neq 0$$

$$H_0(\delta) = A$$

Let δ be a differential and d be an odd derivation.We say that d is a "differential modulo δ" when the following two conditions hold (i) d commutes with δ (in the graded sense) ie

$$d\delta + \delta d = 0$$

(ii) d^2 is δ-exact, ie

$$d^2 = -[\delta, \triangle]\ (\equiv -\delta \triangle - \triangle \delta)$$

for some derivation $\triangle$.

The first condition simply implies that d induces a derivation in $H_*(\delta)$ still denoted by d. The second condition implies that the induced derivation is a differential, namely $d^2 = 0$ in $H_*(\delta)$.

We also state the main theorem of homological perturbation theory (for the proof and for more details see [2])

Theorem: (a) If $\mathcal{H}_k(\delta) = 0$ for all $k \neq 0$, there exists a differential s of total ghost number one (i.e. of total grading one) that combines d with δ,

$$s = \delta + d + \overset{(1)}{s} + \overset{(2)}{s} + ...$$

$$r(\overset{(k)}{s}) = k,\ gh(\overset{(k)}{s}) = 1,\ s^2 = 0$$

(b) Any differential s that combines d with δ as above satisfies

$$H^k(s) = H^k(d)$$

where the cohomology of d is computed in $H_0(\delta)$ (Note: $\mathcal{H}_k(\delta)$ are the induced homology groups for the Lie algebra of derivations).

2.3 DeRham Currents

Given an n dimensional manifold, a "current" is by definition a linear continuous functional $T[\phi]$ defined on the vector space of all C^∞ forms ϕ with compact support. The continuity has the following sense:

If $\phi_h (h = 1, 2, ...)$ is a sequence of C^∞ forms with supports all contained in a single compact set which is in the interior of the domain of a local coordinate system, $(x^1, ..., x^n)$ such that each derivative of each coefficient of the form ϕ_h (represented using $x^1, ..., x^n$) tends uniformly to zero as $h \to \infty$, then $T[\phi_h] \to 0$.

A current T is said to be "homogeneous of dimension p and of odd (even) type" if $T[\phi] = 0$ for each homogeneous form which is not of degree p and even (odd). Each current can be uniquely decomposed as the sum of $2(n+1)$ homogeneous currents, its "homogeneous components". Clearly the notion of currents contains as particular cases both chains and forms, for (i) A chain c in the manifold defines a current

$$c[\phi] = \int_c \phi$$

(ii) A locally integrable form a in the manifold defines a current

$$a[\phi] = \int a \wedge \phi$$

We shall now define boundary operator and differential of a current. Form Stokes' formula we have

$$bc[\phi] = c[d\phi]$$

where b is the boundary operator,d is the differential,c is a chain and ϕ is a form.Thus we can define the boundary bT of a current T by

$$bT[\phi] = T[d\phi]$$

It is clear that bT is well defined and that b maps currents continuous of order p to continuous currents of order $(p+1)$. The "differential dT" of a current T is the current

$$dT = wbT$$

where $w = \pm 1$ defined by the condition that $wT = (-1)^p T$ whenever T is homogeneous of degree p.

Moreover, for each current T we have $b^2 T = d^2 T = 0$.

There is a whole theory dealing with functional analysis and currents. However, we are mostly interest in homology theory of currents. (The general reference for currents is [3]).

A current T is said to be "closed" if $bT = 0$, "homogeneous to zero" if there exists a current S such that $T = bS$ (a different expression for this is "T bounds S"). Two currents are "homologous" if their difference is homologous to zero.

The set of all currents homologous to a closed current form a "homology class", then all homology classes forms an additive group, called "homology group". This group is the quotient of the additive group of closed currents devided by the subgroup of currents which are homologous to zero.

We have also the following properties: (i) each closed current is homologous to a C^∞ form (ii) if C^r-form bounds a current, it bounds a C^r-form.

We can define wedge product of homology classes and thus all the homology groups become a ring. .

We have the property that a current T is homologous to zero iff $T[\phi] = 0$ for all closed forms ϕ. Moreover a current T is closed if it is homologous to a chain.

Hodge theory can also be applied to currents. If we define the "adjoint $*T$" of a current T as

$$*T[\phi] = T[w * \phi]$$

and "scalar product between forms" as

$$(\alpha, \beta) = \int \alpha \wedge *\beta = a[*\beta]$$

then the "scalar product of a current T and a form ϕ" is

$$(T, \phi) = (\phi, T) = T[*\phi]$$

We can define the "codifferential" δ and the Laplacian Δ in the same way as for forms, ie.

$$\delta = - * d * \text{(for an even dimensional manifold)}$$

and

$$\Delta = d\delta + \delta d$$

Then a current T is called “coclosed” of $\delta T = 0$, “cohomologous to zero” if there exists a current S such that $T = \delta s$ and “harmonic” if $\Delta T = 0$

Chapter 3

First Qantized Theory

3.1 Hamiltonian Mechanics

We begin with the "reduced phase method" and we want to find a clear mathematical description of the "observables". We have a phase space P with $\dim P = N$. We consider for simplicity first class primary constraints

$$\phi_m(q,r) = 0, m = 1, \dots, M \tag{3.1}$$

Equations 3.1 define a submanifold Σ smoothly embedded in plase space P, known as a "constraint surface" with $\dim \Sigma = N - M$. P also has a symplectic structure. For convenience we consider the "canonical symplectic form on P". Let M be the configuration space, so $T^*M \equiv P$. The map which assigns to any $\eta \in T^*M$ the contangent vector $\tau^*_{T^*M}(n) \in T^*_\eta T^*M$ defines a horizontal 1-form $\theta \in \Lambda^1_0 \tau^*_M$. Clearly θ is a section of the bundle $((\tau^*_M)^*(T^*M), (\tau^*_M)^*(\tau^*_M), T^*M)$, i.e. the pull back of τ^*_M by τ^*_M and this vector bundle is a vector subbundle of T^*T^*M. If (q_i, p_i) are coordinates on T^*M, then the "canonical 1-form" on T^*M is

$$\theta = p_i \, dq^i \tag{3.2}$$

Its differential $d\theta \in \Lambda^2_1 \tau^*_M$ is called the "canonical symplectic form" on T^*M and we shall denote it by σ. Now σ with the constraint functions ϕ_m define M vectors x^n_m through

$$x^i_m = \sigma^{ij} \partial_j \phi_m \tag{3.3}$$

which are linearly independent at each point of the constraint surface and often called "Hamiltonian vector fields" associated with the functions ϕ_m. They generate infinitesimal gauge transformations. They also generate M dimensional submanifolds on the constraint surface Σ. By construction any vector tangent to these M dimensional surfaces spanned by vector fields x^i_m on the constraint surface is annihlated by the canonical symplectic form on Σ induced by the initial one defined on P and for that reason the surfaces are called "null". But the vector fields x^i_m also generate the gauge transformations, so the null surfaces coincide with the gauge orbits. The first class constraint functions ϕ_m define not only the constraint surface Σ through $\phi_m = 0$ but also the gauge orbits through $\delta_\epsilon F = \epsilon^m (x_m F)$ with $x_m F = [F, \phi_m]$ Poisson bracket and F arbitrary dynamical variable and ϵ^m gauge parameter associated to the first class constraint ϕ_m according to Dirac's conjecture.

Since θ is horizontal, σ is horizontal and since the Hamiltonian vector fields by construction are anhillated by σ, then the vector fields x_m are vertical and the gauge orbits are just the vertical directions on Σ (the Hamiltonian vector fields are tangent to the gauge orbits more generally). A vector field on Σ which is tangent to the gauge orbits is called vertical. Let us denote $\nu(\Sigma)$ the set of vertical vector fields and $\nu^*(\Sigma)$ its dual space, i.e. the "vertical 1-forms". We generalize to consider the verticle p-forms. The exterior product of vertical forms is defined in the standard manner. We shall however modify the wedge product by defining

$$\alpha \cdot \beta = \alpha \wedge \beta (-)^{qp}$$

for a p-form α and a q-form β, where of course "$\wedge$" is the standard wedge product.

We know that the vertical vector fields form an infinite dimensional subalgebra of the infinite dimensional Lie algebra of all vector fields defined on a manifold. Hence we can consider an exterior derivative d acting on vertical forms and taking only antisymmetrised derivatives along the gauge orbits. It is defined formally by the following properties:

- $(dF)(X) = \partial_X F$, where F are functions on a constraint surface and X are vertical vector fields,
- $d^2 = 0$, and
- $d(\alpha\beta) = \alpha\, d\beta + (-)^q (d\alpha)\beta$,

for arbitrary vertical p-form α and q-form β.

Since $d^2 = 0$, we can define cohomology groups as usual

$$H^p(d) = \left(\frac{\mathrm{Ker}\, D}{\Im D} \right)$$

For $p = 0$ we get the gauge invariant functions. So the "observables" are just $H^0(d)$. This is the result we wanted.

In brief, what we really did was to construct a subcomplex of the usual de Rham complex over a manifold considering the submodules of the vertical forms only as modules of the usual complex and the differential d is really the restriction of the usual d to act by taking derivatives along vertical vector fields only. The reason for that is because we want gauge invariant functions., i.e. constant along gauge orbits (which are the vertical directions on Σ). However, another possibility exists: if gauge transformations are mod out by the beginning, i.e. we could define the "reduced phase space" to be a manifold $E = \Sigma/G$, where G is the compact Lie group that gauge transformations form. Then we would have the usual de Rham complex over E. The definitions of the (classical) observables would be the same. This approach is described in [5].

3.2 BRST formalism

We now turn to the BRST formalism

As we indicated above, the central idea of BRST theory is the replacement of the original gauge symmetry "d" by a fermionic rigid symmetry "s" acting on an appropriately extended space containing new variables, the "ghosts". We shall work in the frame of "Pertubative Homological Algebra".

The BRST differential and the complex must be constructed in such a way as to give

$$H^0(s) = \{\textit{Classical observables}\} \tag{3.4}$$

In particular the two steps in the calculation of (classical) observables in the "reduced phase space" method correspond to two auxiliary differentials δ, d in the construction of the BRST differential s.

- The first derivation δ yields a Koszul resolution of the algebra $C^\infty(\Sigma)$. It acts on polynomials in some generators Θ_a with coefficients belonging to $C^\infty(P)$, i.e. the algrbra is $\mathbb{C}[\Theta_a] \otimes C^\infty(P)$. The differential of the complex is denoted by δ. The purpose of the generators Θ_a is to kill the functions of $C^\infty(P)$ that vanish on Σ. That is what we want because as we know $C^\infty(\Sigma) = C^\infty(P)/\mathcal{N}$ where $\mathcal{N}$ is the ideal of $C^\infty(P)$ consisting of functions vanishing on Σ. So the first step of the reduction from $C^\infty(P)$ to $C^\infty(\Sigma)$ is achieved by considering the differential graded algebra $(\mathbb{C}[\Theta_a] \otimes C^\infty(P), \delta)$ which is a resolution of $C^\infty(\Sigma)$, i.e. $H_0(\delta) = C^\infty(\Sigma)$ and δ acyclic in degree $k \neq 0$.

- The second differential d is the vertical exterior derivative d we used in the reduced phase space method which is such that $H^0(d) = \{\textit{classical observables}\}$. This differential is defined on Σ but can be lifted to P. The lifted differential still denoted by d can be extended to the generators Θ_a of the Koszul complex so as to anticommute with δ. One then finds that d^2 vanishes up to δ-exact terms, i.e. d as a differential modulo δ. We must emphasise that although we extend d to act on bigger spaces, its cohomological groups remain unaltered.

So thus all the ingrediants of homological pertubation theory are met. Therefore the main theorem of homological pertubation theory which combines δ and

d with s in such a way that the cohomology of s equals the cohomology of d holds and hence for degree 0 we obtain $H^0(s) = H^0(d) =$ "classical observables".

Let us now present the construction in detail: Since

$$C^\infty(\Sigma) = \frac{C^\infty(P)}{\mathcal{N}} = H_0(\delta) \equiv \frac{(\mathrm{Ker}\,\delta)_0}{(\Im\delta)_0} \tag{3.5}$$

it is natural to define δ so that

$$(\mathrm{Ker}\,\delta)_0 = C^\infty(P) \tag{3.6}$$

and

$$(\Im\delta)_0 = \mathcal{N} \tag{3.7}$$

To achieve 3.6 we simply set

$$\delta z^A = 0 \tag{3.8}$$

$\forall$ phase space variables z^A. But δ acts as a derivation, so $\delta F = 0\ \forall F(z^A)$. If we take the degree of z^A (called the antighost number of z^A) to be zero, then 3.8 implies 3.6.

To obtain equation 3.7 we introduce as many generators Θ_m as the constraints ϕ_m and we take

$$\delta\Theta_m = -\phi_m \tag{3.9}$$

then we get $F \approx 0 \Leftrightarrow F = \delta(-F^m\phi_m) \in (\Im\delta)_0$.

To satisfy the grading properties of δ we take

$$\begin{aligned} \mathrm{antigh}\,\Theta_m &= 1 \\ \epsilon(P_m) &= \epsilon_m + 1 (\epsilon_m \equiv \epsilon(\phi_m) \textit{the parity}) \end{aligned}$$

Then we extend δ to an arbitrary polynomial in Θ_m's with coefficients that are phase space functions, i.e. to the algrbra $\mathbb{C}[\Theta_m] \otimes C^\infty(P)$ by requiring δ to be an odd (right) derivation. Since δ^2 vanishes on all generators, it is nilpotent. So with 3.8 and 3.9 condition 3.5 is fulfilled. The variables Θ_m are called the "ghost momenta".

One can then prove that the differential algrbra $(\mathbb{C}[\Theta_m] \otimes C^\infty(P), \delta)$ provides a resolution of $C^\infty(\Sigma)$ (for the proof see [2] p198) and hence the first part of the construction is done.(In the sequel we shall call this resolution the "Koszul-Tate resolution")

One can prove for the irreducible case that the exterior algebra of vertical forms is isomorphic to $C^\infty(\Sigma) \otimes \mathbb{C}[\omega^a]$ where ω^a are the 1-forms dual to the vector fields which define the infinitesimal gauge transformations (these are just the Hamiltonain vector fields and they are vertical). These 1-forms are the "ghosts" and we shall denote them by η^a. The form degree is the "pure ghost number". We have

$$\begin{aligned} \text{puregh}\, \eta^a &= 1 \\ \text{puregh}\, z^A &= 0 \end{aligned}$$

We note that although the action of d is defined on the whole algebra $C^\infty(P) \otimes \mathbb{C}[\eta^a]$, it is not a differential in $C^\infty(P) \otimes \mathbb{C}[\eta^a]$ because the square is only weakly zero.

We observe that the Koszul-Tate complex and the exterior vertical complex possess exactly the same number of generators because it is the same (first class) functions ϕ_m that generate them both. So it is natural to consider the ghosts η^a as being conjugate to the ghost momenta Θ_a and to extend the (Poisson) bracket structure from the original phase space P to the space $\mathbb{C}[\Theta_a] \otimes C^\infty(P) \otimes \mathbb{C}[\eta^a]$, called "extended phase space" and denoted by P_{ext}, as follows

$$\begin{aligned} \left[\Theta_a, \eta^b\right] &= -(-\epsilon)^{(\epsilon a+1)(\epsilon b+1)} \\ \left[\eta^b, \Theta_a\right] &= -\delta^b_a \end{aligned}$$

and taking brackets of η^a and Θ_a with original variables to be zero and brackets of original variables among themselves unchanged.

Now the final step is to bring δ, d together, because they have been defined on $\mathbb{C}[\Theta_a] \otimes C^\infty(P)$ and on $C^\infty(P) \otimes \mathbb{C}[\eta^a]$ respectively. So we extend δ to the whole of the P_{ext} by setting

$$\delta\eta^a = 0$$

and d to P_{ext} by setting

$$d\Theta_a = (-)^{\epsilon a} \eta^c C^b_{ca} \Theta_b$$

This definition yields $[\delta, d] = 0$. Furthermore, d^2 is δ-exact so d is a differential monulo δ

Because the ghosts η^a are δ-closed but not δ-exact, the homology of δ in the algebra P_{ext} is just tensor product of homology of δ in $\mathbb{C}[\Theta_a] \otimes C^\infty(P)$ with the ghosts, i.e.

$$\begin{cases} C^\infty(\Sigma) \otimes \mathbb{C}[\eta^a] \quad \text{for } k = 0, \\ \text{acyclic} \\ \text{otherwise.} \end{cases}$$

So δ porvides a resolution of the exterior vertical algebra $C^\infty(\Sigma) \otimes \mathbb{C}[\eta^a]$. The cohomology of d modulo δ coincides then with the cohomology of the vertical exterior derivative defined on Σ.

The same construction appears in [5]. The only difference is the the extended phase space is replaced by

$$\Lambda(g \oplus g^*) \otimes F(P) = \Lambda(g) \otimes \Lambda(g^*) \otimes F(P)$$

where $F(P)$ is the ring of functions defined on the manifold P which is the phase space, g and g^* are the Lie algebra and its dual corresponding to the Lie group that the gauge transformations form and $\Lambda(g)$, $\Lambda(g^*)$ their exterior algebra (considering g and g^* as vector spaces). This is more or less the same thing, since first class constraints are the generators of gauge transformations.

However, we make the following comment: because the final step is to quantize the system, the expression of s as a differential is not a convenient one. For instance if we could represent s by an element of the extended algebra $\mathbb{C}[\Theta_a] \otimes C^\infty(P)$, say Ω, and if transformation of an arbitrary function F could be written as a canonical transformation, i.e.

$$sF = [F, \Omega]$$

that would be very helpful because the quantization process by using canonical commutation relations would be straightforward.

The answer is of course that all these can be done. Such an element Ω can be found and it is called the "BRST generator". The nilpotency of s is translated as

$$[\Omega, \Omega] = 0$$

and this is the cornerstone of BRST symmetry.

We close this section with a final comment: the formula

$$s = \delta + d + \text{ “more”}$$

reminds one of the definitions of the total differential in a double complex. This is indeed also the case here, we could have considered a double complex consisting of two differential graded algebras. The result would be that classical observables would be the $(0,0)$ degree group of the resulting spectral sequence. This approach is more or less the same. We do not discuss it here. More details can be found in [5].

3.3 First Quantization

We now turn to the quantum level. For the reduced phase space method, we have not much to say.We must find a Hilbert space on which some operators act. By the quantization process these operators are just the classical observables, i.e. functions on phase space constant along the gauge orbits.

In the BRST formalism the ghosts and their conjugate momenta are also realised as linear operators, as well as the BRST generator Ω (sometimes also called "BRST charge"). The Poisson bracket has become commutor (graded). We still want

$$[\hat{\Omega}, \hat{\Omega}] = 0 \tag{3.10}$$

for the operator $\hat{\Omega}$ and this comes from the nilpotency of Ω. We also want

$$\hat{\Omega}^* = \Omega(\text{Hermitian}) \tag{3.11}$$

because Ω was a real function in the classical level.

We can also define the "BRST (quantum) observables" as the operators satisfying

$$[\hat{A}, \hat{\Omega}] = 0$$

Now we can define the "Quantum BRST operator cohomology" by taking

$$[\hat{A}, \hat{\Omega}] = 0 \Leftrightarrow \hat{A}\text{is closed}$$

and

$$\hat{A} = [\hat{B}, \hat{\Omega}] \Leftrightarrow \hat{A}\text{is exact}$$

where $[,]$ is the graded commutator and define "Quantum operator cohomology groups" as closed modulo exact operators.

However, in the quantum level, there are also "state cohomology groups". In particular, the nilpotency of $\hat{\Omega}$ implies that any state of the form $\hat{\Omega}x$ obeys

$$\hat{\Omega}(\hat{\Omega}x) = \hat{\Omega}^2 x = 0$$

Since in the classical level the whole Koszul-Tate resolution construction was such that the BRST and the gauge invariant (= classical observables) functions coincide (that was the meaning of demanding zero cohomology of s = zero cohomology of d (= classical observables)), we must also impose the "reality condition"

$$\hat{\Omega}x = 0 \tag{3.12}$$

The "classical BRST observables" are $[\Omega, F] = 0$ and by construction coincide with gauge invariant functions on P. Because we extended the phase space to P_{ext}, the corresponding Hilbert space in the quantum level would include ghosts as other states which are unphysical and therefore we want to mod them out. The above reality condition does this for us.

Now we can also define state cohomology groups as follows:

$$\hat{\Omega}\psi = 0 \Leftrightarrow \psi \text{ is closed}$$

and

$$\psi = \hat{\Omega}x \Leftrightarrow \psi \text{ is exact}$$

So quantum state cohomology is closed modulo exact states.

We finish this section with some comments.

We must emphasise that although in the classical level we know a priori that a BRST generator existed in order to interpret the BRST transformations as canonical transformations, in the quantum level this is no longer the case. In other words, because of the question of ordering of the factors which comes in crucially in the quantum level, it is not granted that a linear operator satisfying 3.10 or 3.11 exists, even if the BRST generator Ω exists in the classical level.

Moreover, although we do not discuss these problems here, the quantization process is by no means accomplished. In addition to the above problem, there is also a problem in defining the physical Hilbert space, i.e. problems concerning equation 3.12, the "reality condition". Another important problem finally that may arise in the quantum level are the "anomalies". We refer to [2] for more details.

Chapter 4

Field Theory

4.1 Classical Field Theory

In Field Theory(FT), as it is well known, the notion of a particle is replaced by the notion of a field. This enables one to proceed directly to the second quantized theory in the quantum level using the BRST formalism. The mathematical description of FT is as follows:

1. Matter fields ϕ are represented by global sections of a vector bundle E associated with a principal bundle P whose structure group is G (If particles are fermions,we use spin bundles,if they are bosons we use real vector bundles).Fibres are the gauge orbits.

2.Principal connections on P are identified with gauge potentials.To be more precise,the gauge potentials are really the pull-backs of the principal connections by a global section of P.Changing of the trivialisation or equivalently moving to an isomorphic bundle means gauge transformation (of second kind).The choice of a section means fixing the gauge.

3.The base space of both P and E is the space-time manifild X,assumed to be four dimensional,oriented and connected.(In Physics,that is in QFT, it is also usually assumed to be flat)

4.Since we are considering,as this is always the case in Physics,only first variations of the fields,a configuration space of the fields ϕ is the first jet manifold J^1E and their momentum space is the Legendre bundle Π over E (to be defined in the sequel).Similarly,for the gauge potentials (here there are some subtleties which will be considered later,because we shall use a somehow different,more convenient in our case,approach to describe the space of all connections (for this being mathematically legitimate,c.f.[1] p.146))

Although for simplicity we shall not consider in this piece of work this situation,

5.Spontaneous symmetry breaking with an exact symmetry subgroup H of the structure group G will give rise to Higgs fields which are global sections of the quotient bundle $\Sigma = P/H$.Given an atlas Ψ^{Σ} of the bundle Σ, the field components $(\psi_k^{\Sigma} h)$ of a Higgs field h admit the decomposition:

$(\psi_k^{\Sigma} h)(x) = h_k(x)\sigma_0$, $h_k(x)G(U_k)$

where σ_0 is the H-stable center of the quotient space G/H and $G(U_k)$ is the group of all the G-valued functions defined on U_k (where U_k is open in X).The fields $h_k(x)$ in the decomposition are the Goldstone bosons.(c.f.[4] for more details).We just mention here that the Higgs fields fail to form a vector or an affine space.

6.A Lagrangian L of fields ϕ on J^1E is required to be gauge invariant.This will be modified in the BRST formalism by imposing the condition that L should satisfy the same equations of motion(and that weaker condition will give us the gauge transformations of the Lagrangian itself).

If we denote the principal bundle P by (P, π_P, X),then the associated vector bundle E will be denoted by (E, π, X).We shall briefly refer to a bundle by its total space or by its projection only.

The principal connections on the principal bundle P can be represented by G-equivariant sections of the bundle $(\pi_P)_{1,0}$.Consequently,there is a 1:1 correspondence between principal connections A on P and global sections A^C of the affine bundle

$C = J^1P/G (= J^1P/G \to P/G = X)$, where the projection of this bundle will be denoted by π_C , modelled over the vector bundle

$$\bar{C} = T^*X \otimes V^G P$$

(where $V^G P$ will be defined in the sequel). We call C the connection bundle.Hence the gauge potentials will be the global sections A^C of the connection bundle C (to be more precise, their pull-backs by a section of the bundle π_P).So the configuration space of the gauge potentials is just the first jet manifold J^1C.The affine bundle J^1C admits the canonical splitting:

$$J^1C = C_+ \oplus_C C_- = (J^2P/G) \oplus_C (\wedge^2 T^*X \otimes V^G P)$$

where C_+ is the affine bundle modelled over the vector bundle

$$\bar{C}_+ = S^2T^*X \otimes V^G P$$

where S denotes the symmetric product.The second projection pr_2 of the splitting of J^1C defines the fundamental form

$$\mathcal{F} : J^1P \to \wedge^2 T^*X \otimes V^G P$$

If A is a principal connection on P ,its curvature form is just

$$\mathcal{F}_A = \mathcal{F} j^1 A^C$$

and this is all we need to construct gauge invariant Lagrangians out of gauge potentials,whereas the form

$$s : J^1C \to C_+$$

is defined by gauge conditions.(Note: in the above disscusion,V^GP denotes the vertical sub-bundle VP of TP devided by the structure group G and $\wedge$ is just the exterior product).

We do not assume,for simplicity, any space-time symmetries.However,we shall comment in the end of this section,on the neccessary modifications in our construction needed if one considers them as well.

We have then the result that the "total" configuration space is

$$J^1E \times J^1C = \Xi$$

A "Lagrangian density" λ on a bundle (E, π, X) is by definition a function of the algebra $C^\infty(J^1\pi)$.The corresponding "Lagrangian" is then the m-form $L = \lambda\Omega$ belonging to the set $\wedge_0^m\pi_1$ (i.e. it is a horizontal m-form).

In our case,we have a four dimensional space-time manifold, so the Lagrangian L belongs to the set $\wedge_0^4\pi_1$ for the matter part of the total Lagrangian and the gauge part of the total Lagrangian belongs to the set $\wedge_0^4(\pi_C)_1$. (Note: We identify the volume form of the space-time manifold with its pull-back by π_1 on J^1E and,similarly for the connection bundle).

We now wish to describe the Legendre bundle Π over E,and analogously for the bundle C.The global sections of this bundle will correspond to the conjugate momenta of the fields.(For a more detailed description we refer to [4]).

Let us assume that we have the vector bundle (E, π, X). Then by definition, one calls the vector bundle

$$\Pi = \wedge^n T^*X \otimes TX \otimes_E V^*E$$

over E the "Legendre bundle".This is provided with the so called standard bundle coordinates $(x^\lambda, y^i, p_i^\lambda)$ such that

$$\pi \circ \pi_\Pi : \Pi \to E \to X$$

and

$$\pi \circ \pi_\Pi : (x^\lambda, y^i, p_i^\lambda) \to (x^\lambda, y^i) \to (x^\lambda)$$

The Legandre morphism

$$\hat{L} : J^1E \to \Pi$$

is just the fibre derivative of L.

We can define the action S of L as usually, by integration. It is a de Rham current defined on the exterior algebra of forms $\wedge\Xi$ (clearly the full Lagrangian density is a function of $C^\infty(\Xi)$ and it is simply the sum of the matter field and the gauge part).

Our physical requirement that S is stationary, leds to the Euler-Lagrange equations for e

$$(j^2 e)^*(\delta L) = 0 \tag{4.1}$$

where e is a section of E and δ is the Euler-Lagrange operator.

Proposition:
The Euler-Lagrange equations can be written as

$$dL_0 = 0$$

where

$$L_0 = L + \Theta_L$$

and Θ_L is the Cartan form of L.

Proof:

By definition, the Euler-Lagrange form δL of L is

$$\delta L = (\frac{\partial \lambda}{\partial u^a} - \frac{d}{dx^i}\frac{\partial \lambda}{\partial u^a_i})du^a \wedge \Omega$$

Let us assume that the base manifold X has dimension m and we consider the bundle (E, π, X).Then L is an m-form on $J^1\pi$ and it can be written as

$$\delta L = \pi^*_{2,1} dL + d_h \Theta_L$$

Of course, dL is an (m+1)-form on $J^1\pi$ and hence $\pi^*_{2,1}dL$ is an (m+1)-form on $J^2\pi$.Moreover, Θ_L , the Cartan form of L is an m-form on $J^1\pi$.The differential d_h carries m-forms of $J^1\pi$ to (m+1)-forms of $J^2\pi$,i.e. by definition it is a chain homotopy between the two deRham complexes over the manifolds $J^1\pi$ and $J^2\pi$ and because the "pull-back" of forms is a linear operator which commutes with the differential d (see for instance [3]) we then have that

$$d_h = \pi^*_{2,1} d$$

and the order of writing the operators in the composition makes no difference since, as we just said, they commute.(Strictly speaking, there are two differentials d, one defined on $J^1\pi$ and one defined on $J^2\pi$ but we use the same symbol for both of them.)

Hence , by substituting the last equation to the previous one, we obtain

$$\delta L = \pi^*_{2,1} dL + \pi^*_{2,1} d\Theta_L$$

But since the "pull-back" operator is also linear, then we obtain

$$\delta L = \pi_{2,1}^*(dL + d\Theta_L) = \pi_{2,1}^* d(L + \Theta_L) = \pi_{2,1}^* dL_0$$

(the exterior differential d is also linear).We have set $L_0 = L + \Theta_L$

We now substitute the last equation to the initial Euler-Lagrange equations for the section e,

$$(j^2 e)^*(\delta L) = 0$$

and we obtain

$$(j^2 e)^* \pi_{2,1}^* dL_0 = 0$$

But $j^2 e$ is a second order jet field, i.e. a section of the bundle $\pi_{2,1}$ (the second prolongation of a section e of the bundle π) and hence we have:

$$(j^2 e)^* \pi_{2,1}^* dL_0 = 0 \Longleftrightarrow dL_0 \pi_{2,1} j^2 e = 0 \Longleftrightarrow dL_0 = 0$$

for $j^2 e$ section of $\pi_{2,1}$
The general property

$$\theta^* \pi^* L = L \pi \theta$$

($= L$ iff θ a section of π) can also be represented ,in arbitrary notation, via the following commutative diagram:

$$\begin{tikzcd}
M \times E \times T^*M \arrow[r] & E \times T^*M \arrow[r] & T^*M \\
M \arrow[u, "\theta^*\pi^*L"] \arrow[r, "\theta"'] & E \arrow[u, "\pi^*L"] \arrow[r, "\pi"'] & M \arrow[u, "L"']
\end{tikzcd} \tag{4.2}$$

We have then:

$$\pi^* L = L \circ \pi \Longrightarrow \theta^* \pi^* L = \theta^*(L \circ \pi) = L \circ \pi \circ \theta = L \circ 1 = L$$

Q.E.D.

We should also mention something about the Cartan form Θ_L of L:

Θ_L is completely and uniquely determined by L itself via the following expression (in local coordinates):

$$\Theta_L = \frac{\partial \lambda}{\partial u_i^a}(du^a - u_j^a dx^j) \wedge (\frac{\partial}{\partial x^i} \lrcorner \Omega) + L$$

In the sequel we shall use both L and L_0 as the Lagrangian.

Now we wish to interpret the fact that L must be gauge invariant.

We begin by recalling that the isomorphisms Φ_P of the principal bundle P induce associated isomorphisms Φ_E of the associated vector bundle E. We call the former "principal isomorphisms" and the latter "associated principal isomorphisms".

We claim that the requirement that L is invariant under the maps

$$j^1\Phi_E = j^1\hat{P}_s, \quad \hat{P}_s = \hat{P}_E|_{E\times s(X)}$$

for all global sections s of the principal group bundle $\tilde{P}$, is a necessary and sufficient condition for L to be gauge invariant.

(**Note:** A group bundle is defined to be a fibre bundle , say E , together with canonical bundle morphisms m and k over the base manifold , say B , and a global section e_E of E :

$$m : E \times E \to E$$

$$k : E \to E$$

$$e_E : B \to E$$

They make each fibre $E_x = \pi^{-1}(x)$ of E into a Lie group as follows:

$$m(e_E(x), y) = m(y, e_E(x)) = y$$

$$m(k(y), y) = m(y, k(y)) = e_E(x), y \in E_x$$

Now the principal group bundle $\tilde{P}$ of a principal bundle P is the P-associated bundle with the standard fibre G on which the structure group G acts by the adjoint representation.Fibres of $\tilde{P}$ are isomorphic (but not canonically isomorphic) to the structure group G.We recall that fibres of the principal bundle P are diffeomorphic to the group G but fail to be groups.Moreover , for any P-associated bundle E , the canonical bundle morphism

$$\hat{P}_E : E \times_B \tilde{P} \to E$$

is defined.)

We can replace global sections of $\tilde{P}$ by its local sections and in turn, by their jet prolongations. So, we require L to be invariant under the following jet prolongations

$$j^1P_E : J^1\tilde{P} \times J^1E \to J^1E$$

of the canonical maps

$$P_E : \tilde{P} \times E \to E$$

This means that

$$L(w) = L(j^1\hat{P}_E(q, w)) \tag{4.3}$$

for each $w \in J^1E$ and $q \in J^1\tilde{P}$.

(For the proof of the claim see [4] and references therein).

In order to find a more explicit formula for 4.3, we firstly observe that there is a 1:1 correspondence between generators of infinitesimal principal isomorphisms and right invariant sections of the bundle $V\tilde{P}$ (vertical subbundle of $T\tilde{P}$). Moreover, every local right invariant section of $V\tilde{P}$ induces a local vertical field on the bundle E, called principal vertical vector field. Hence, we have a 1:1 correspondence between the generators of infinitesimal principal isomorphisms and principal vertical vector fields, which are the gauge generators u_g. But in order for u_g to act on L, we must consider their verical lifts on the jet manifold J^1E, denoted by $\bar{u}_g$. Then the gauge invariance of the Lagrangian L reads

$$\mathcal{L}_{\bar{u}_g} L = 0 \tag{4.4}$$

where $\mathcal{L}$ is the generalised Lie derivative given by the Frölicher-Nijenhuis (F-N) bracket. We recall that tangent valued 0-forms (i.e. vector fields) with the usual commutation brackets form a sheaf which can be identified with the sheaf of Lie algebras. The sheaf of tangent valued forms endowed with the (F-N) brackets can be identified with the sheaf of graded Lie algebras.

If we wish to take into account space-time symmetries as well,then we should consider the group $Diff_X P$ of "general principal isomorphisms",i.e.

$$\Phi_P(p, g) = \Phi_P(p)g$$

where p is in P and g is in G,of the principal bundle P.

By using similar arguments as above,we observe that we have a 1:1 correspondence between the generators of infinitesimal general principal isomorphisms and principal vector fields u (i.e. projectable vector fields of E,and not vertical vector fields as in the previous case).Denoting by $\bar{u}$ the lift of u (not the vertical lift as in the previous case) onto the jet manifold J^1E we obtain the relation

$$\mathcal{L}_{\bar{u}} L = 0$$

which is the gauge invariance condition for the Lagrangian, in the presence of space-time symmetries.

Both the above gauge invariance conditions can give us the Nöther identities.(for more details,see [4])

If we impose the more general condition that the Euler-Lagrange equations remain unaltered, we obtain the gauge transformations of L_0, i.e. if we add a d-exact term to L_0 , then the Euler-Lagrange equations remain the same.This is very straightforward from the form of the Euler-Lagrange equations that we have presented here.Clearly, the d-exact term added to L_0 is the gauge-fixing term of the Lagrangian.

Let us now examine the action itself.Since both chains and cochains can be considered as currents, it is clear the action can be thought of as a current itself.Namely,

$$S = C[L] = \int_C L$$

(since Θ_L is completely determined by L, we identify L with L_0).
The condition that S is stationary gives the Euler-Lagrange equations

$$dL = 0$$

Clearly, from the definition of the closed currents,this immediately implies that S is closed,i.e.

$$bS := C[dL] = 0$$

This is the analogue,in our presentation, of the common functional equation for the action:

$$\frac{\delta S}{\delta \phi_i} = 0$$

where the ϕ_i's are the fields appearing in the Lagrangian.

We can now simply say that the gauge transformations of the action are simply:

$$S \to S + bT$$

where T is an arbitrary current.

Let us now consider two cases:

1."Physical Theories" In this case, the space-time manifold is always the same manifold X and we can take different theories by considering different Lagrangians, i.e. the action can be essentially identified with the Lagrangian,which is of course a form(cochain).Hence the gauge transformations of the action are simply the gauge transformations of the Lagrangian (and since it is a form, it is just the addition of a d-exact term, i.e. the gauge-fixing term, as we have already described previously)

2."Topological Field Theories" In this case, we are dealing with manifolds with boundary (space-time manifold is not fixed, neither is the Lagrangian of course)(see for instance [6]), and hence we can have , roughly speaking, two kinds of gauge transformations:

a.The first refers to the Lagrangian (cochain) which is the same as in case 1. above

b.The second refers to the chain, i.e. the domain we intergrate in order to obtain the action.In this case, (keeping the Lagrangian fixed) we can essentially identify the action S with the chain C and hence the gauge transformations of the action simply read

$$C \to C \amalg \partial D$$

where D an arbitrary chain.Clearly, this gives us the cobordisms of C modulo some ambiguity concerning the orientation.(see for instance [7])

4.2 Quantum Field Theory

In order to quantize the theory , we shall imitate the steps leading to the first quantized theory.

4.2.1 Reduced Phase Space Method

The reduced phase space method in this setting is modified as follows:

i.) To the initial phase space P corresponds the manifold $\Xi \times \Xi^L = \tilde{\Xi}$, i.e. the total configuration space of the fields and their conjugate momenta.

ii.) To the constrained surface Σ corresponds the submanifold $\tilde{\Xi}_\Sigma$ of the total configuration space $\tilde{\Xi}$ defined by the constraints' equations.

iii.) To the reduced phase space then just corresponds the manifold $\tilde{\Xi}^G_\Sigma = \tilde{\Xi}_\Sigma/G$ (Note:Because by definition the bundle C is J^1P/G , the structure group G should be mod out only from the J^1E factors of the total configuration space.)

For the quantization procedure, one replaces as usual the Poisson bracket by the commutator of the corresponding operators.(We still need a symplectic form defined on E in order to have the induced symplectic form defined on the total configuration space, just like in the first quantized theory case)

4.2.2 BRST method

Let us now examine the BRST formalism.The usual differential d acting on forms of the space-time manifold X (and hence $H^0(d)$ is just the set of the classical observables) is now replaced by the differential $s = d + \delta$, where d is the same as before and δ is the differential of a resolution of the algebra $C^\infty(P)$, exactly as before.

We use the standard way to add odd variables , i.e. by tensoring with $\wedge(g \oplus g^*)$, where g is the Lie algebra of the structure group G and g^* its dual (g is the space of the ghosts and g^* is the space of the ghost momenta.)

So the algebra over the extended phase space we had before $\mathbb{C}[\nleq_{\supset}] \otimes \mathbb{C}^\infty(\mathbb{P}) \otimes \mathbb{C}[\eta_{\supset}]$ is now replaced by the following: $C^\infty(\tilde{\Xi}) \otimes \wedge(g \oplus g^*) = C^\infty(\tilde{\Xi}) \otimes \wedge g \otimes \wedge g^*$

The construction is the same . We obtain the original gauge invariance in the 0th cohomology group , i.e. as usual

$$H^0(s) = H^0(d)$$

We must however notice that one should be careful when generalising the Poisson bracket structure originally on $C^\infty(\tilde{\Xi})$ to the induced Poisson bracket structure on the extended algebra.The brackets between original fields ϕ_i with the corresponding "ghosts" ϕ_i^* is what is called in Physics texts "antifield formalism" (for a more explicit presentation ,which anyway immitates exactly what we did in the first quantized theory case, see [2] p.366 and elsewhere)

We shall finish this part by mentioning that the term "master field" used commonly in Physics texts refers to a field S acting as a generator (or BRST charge) for the BRST transformations, i.e. instead of finding the differential s we find a field S such that Poisson bracket with S gives the BRST transformations. The equation

$$(S, S) = 0$$

which is the equivalent of the nilpotency of s is known as the "master equation" (Recall what we mentioned before about trying to describe the BRST transformations as canonical transformations by using Ω .That was for the first quantized theory.For the second quantized theory we have a field S instead of a phase space function Ω and (,) is the generalised Poisson bracket structure).

Similarly we have the "quantum master equation" which is an operator (remember:unlike the classical situation, this does not neccessarily exist!).We also recall that S (and Ω) are unique up to gauge (canonical) transformation.

The final comment is that one could use the path integral approach instead of the commutation relations in order to carry out the quantization of the system.There are technical differences but the BRST general scheme remains unaltered.After all , the subtleties of the quantization proceedure itself are not dealt with ,except some comments at the end of Chapter 3 , in this piece of work.

Chapter 5

String Field Theory

In this chapter, we shall briefly present how some of the previously described ideas apply to string field theory.What we shall be interested in more specificaly, is the BRST field theory of the interacting open bosonic strings.(The reason for this is that the BRST cohomology brakes down when dealing with closed strings or superstrings).However, the main ideas of this approach are simply an example of Cyclic (Co)Homology. We start by giving a brief account of Cyclic cohomology and Non commutative differential geometry.

5.1 Cyclic Cohomology

We shall start this section with an important definition:

Definition 1 A K-cycle (or Fredholm module) on an algebra $\mathcal{A}$ is given by a unitary representation of $\mathcal{A}$ in a Hilbert space H and an unbounded self-ajoint operator D in H such that:

1. $[D, a]$ is bounded $\forall a \in \mathcal{A}$
2. $(1 + D^2)^{-1}$ is compact.

If $\mathcal{A}$ does not have an identity then we replace 2. by $a(1 + D^2)^{-1}$ compact for all $a \in \mathcal{A}$.

The above definition comes from the Hilbert space interpretation of the Grothendieck group $K(X)$, where X is a topological space.Namely, given a Hilbert space H (seperable,complex), then the set $G(H)$ of all bounded operators on H equipped with the norm topology , makes G into a Banach algebra. An operator T in G is called a Fredholm operator if both its kernel and cokernel are finite dimensional, the difference of the dimensions being the index of the operator.

If we denote by $[X, \mathcal{J}]$ the set of homotopy classes of mappings $X \to \mathcal{J}$, then the main result is that there is a natural isomorphism

$$index : [X, \mathcal{J}] \to \mathcal{K}(\mathcal{X})$$

This means that $\mathcal{J}$ is a classifying space or representation space for K-theory. (Note: the group $\mathcal{B}^*$ is another classifying space for $K(X)$, i.e. $[X, \mathcal{B}^*] \cong \mathcal{K}(\mathcal{X})$

where $\mathcal{K}$ is the set of all compact operators on H , $\mathcal{B}$ is the quotient $G/\mathcal{K}$ and $\mathcal{B}^*$ is the group of units of $\mathcal{B}$.

For more details , see for instance "K-Theory",by M.F.Atiyah , Benjamin 1967)

Definition 2 Let $d \in [1, \infty)$ be a real number. A K-cycle (H, D) on $\mathcal{A}$ is d^+ summable if the eigenvalues E_n of D arranged in increasing order satisfy:

$$\sum_{i=1}^{N} E_n^{-1} = O(\sum_{i=1}^{N} n^{-1/d})$$

For $d = 1$ this means that $\sum_{i=1}^{N} E_n^{-1} = O(logN)$ and for $d > 1$ that $E_n^{-1} = O(n^{-1/d})$. This condition gives an upper bound for the dimension of the K-cycle.

As an example , we may say that the Dirac operator on a spin manifold defines a K-cycle on the algebra $\mathcal{A}$ of functions on the manifold.

The starting point of cyclic cohomology is the possibility, given a K-cycle (H, D) on an algebra $\mathcal{A}$ (and a $Z/2$ grading γ in H such that $\gamma a = a\gamma$ for $a \in \mathcal{A}$, $\gamma D = -D\gamma$), of imitating the differential and integral calculus of differential forms on a manifold in the following manner. The sign $F = D|D|^{-1}$ of the operator D (supposed to be invertible for simplicity) enables us to define an operator-valued differential $da = i[F, a]$ for $a \in \mathcal{A}$, and the property $F^2 = 1$ shows that $d(da) = 0$ (where for $X \in \mathcal{L}(\mathcal{H})$, $X\gamma = -\gamma X$ we put $dX = i(FX + XF)$ instead of $FX - XF$). We then get the complex

$$\mathcal{A} \xrightarrow{d} \Omega^1(\mathcal{A}) \xrightarrow{d} \Omega^2(\mathcal{A}) \xrightarrow{d} ... \tag{5.1}$$

where $\Omega^k(\mathcal{A})$ is the vector space of linear combinations of operators of the form $\omega = a^0da^1...da^k$, $a^j \in \mathcal{A}$

The composition of operators in H allows one to define the product of quantized forms, for $\omega_1 \in \Omega^{k_1}$, $\omega_2 \in \Omega^{k_2}$, one has $\omega_1\omega_2 \in \Omega^{k_1+k_2}$, $d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{\delta\omega_1}\omega_1 d\omega_2$. If the K-cycle (H, D) is d^+ summable, the quantized forms ω in $\Omega^k(\mathcal{A})$ have the following property:

$$\sum \mu_n(\omega)^p < \infty$$

for all $p > d/k$

This property is sufficient for defining the analogue of the integral for the quantized forms $\omega \in \Omega^d(\mathcal{A})$ (assuming d is an even integer) by the equation:

$$\int \omega = Trace(\gamma\omega)$$

This formula does not make any sense as such, since for $\omega \in \Omega^d$ one has $\sum \mu_n(\omega)^p < \infty$ only for $p > 1$, but one can regularise it very simply by replacing

$Trace(\gamma\omega)$ by $Tr_s(\omega) = (1/2)Trace(\gamma F(F\omega + \omega F))$ .As $F\omega + \omega F \in \Omega^{d+1}$ there is no problem whatsoever.One then proves the analogue of Stokes law:

$$\int d\omega = 0$$

for all $\omega \in \Omega^d(\mathcal{A})$, as well as the following property of commutativity under the integral sign:

$$\int \omega_1\omega_2 = (-1)^{\delta_1\delta_2} \int \omega_2\omega_1$$

where ω_1 and ω_2 are of degree δ_1 and $\delta_2$respectively and $\delta_1 + \delta_2 = d$

The notion of a cyclic cocycle τ on an algebra $\mathcal{A}$ is analogous to that of a trace. A trace τ_0 on $\mathcal{A}$ is a linear functional on $\mathcal{A}$ which satisfies the following property:

$$\tau_0(x^0x^1) - \tau_0(x^1x^2) = 0, \forall x^0, x^1 \in \mathcal{A}$$

A cyclic n-cocycle τ_n on an algebra $\mathcal{A}$ is an (n+1)-linear functional on $\mathcal{A}$ satisfying:

$$\tau(x^0x^1, ..., x^{n+1}) - \tau(x^0, x^1x^2, ..., x^{n+1}) + ...$$
$$... + (-1)^j\tau(x^0, ..., x^jx^{j+1}, ..., x^{n+1}) + (-1)^{n+1}\tau(x^{n+1}x^0, ..., x^n) = 0$$

and

$$\tau(x^1, ..., x^n, x^0) = (-1)^n\tau(x^0, ..., x^n)$$

$\forall x^j \in \mathcal{A}$

The homological information given by a K-cycle (H, D) on $\mathcal{A}$ as above is captured by its character which is the following cyclic d-cocycle:

$$\tau(x^0, x^1, ..., x^d) = \int x^0dx^1...dx^d = (-1)^{d/2}Tr_s(x^0[F, x^1]...[F, x^d])$$

If $\mathcal{A}$ is the algebra of functions on the manifold say M, a closed deRham current C of dimension $n \in 0, 1, ..., dimM$ defines a cyclic cocycle τ_C on $\mathcal{A}$ by the equation:

$$\tau_C(f^0, ..., f^n) = \langle C, f^0df^1 \wedge ... \wedge df^n\rangle, \forall f^j \in \mathcal{A}$$

and the construction of the Chern Character:

$$ch : K^*(M) \rightarrow H^*(M, \mathbb{C})$$

which, using connections and curvatures , assigns to each complex vector bundle E over M a cohomology class $ch(E)$ with complex coefficients , is a particular case of the following result , which no longer requires the commutativity of $\mathcal{A}$:

"Let $\mathcal{A}$ be an algebra and τ_{2m} a cyclic cocycle on $\mathcal{A}$. The following equation defines an additive map from $K_0(\mathcal{A})$ to $\mathbb{C}$:

$$\langle e, \tau_{2m} \rangle = \tau_{2m}(e, ..., e) \forall e \in Proj(M_k(\mathcal{A}))$$

where $M_k(\mathcal{A})$ are the $k \times k$ square matrices with entries from the algebra $\mathcal{A}$.

(For more details see "Cyclic Homology" by J-L.Loday , Springer 1988 which is the reference book , although all the material used in this little piece of work can be found in [8]).

5.2 String Field Theory

5.2.1 Light Cone Gauge

We start our discussion by considering the field theory of the open bosonic strings in the Light Cone Gauge(LCG).The reason for this is that this case is very similar to the point particle case.

Let us recall that the Lagrangian for the first quantized theory for the free open bosonic string is just the Nambu-Goto Lagrangian (a more general action for a d-dimensional extended object was given originally by P.A.M. Dirac , the case where $d = 1$ is just the Nambu-Goto action)

$$L = \frac{1}{2\pi a'} \sqrt{\dot{X}_\mu^2 (X'^\mu)^2 - (\dot{X}_\mu X'^\mu)^2}$$

where $X(\sigma, \tau)$ is the string variable , the dot represents differentiation with respect to τ (proper time) and the dash represents differentiation with respect to σ (the string length parameter).This action simply gives the area of the two-dimensional surface (known as the "world-sheet") swept out by the string.The index μ refers to space-time manifold in which the world-sheet is embedded and for which we assume that it is flat with Minkowski signature.The dimension is fixed from other arguments to be 26 , or 10 if we assume supersymmetry.There are also two more formulas for the Lagrangian (the one mentioned above is the "non-linear" one) , the "Polyakov" and the "Hamiltonian" ones , which classically are all equivalent.(For more details see for instance [9]).

If we choose the LCG then the Nambu-Goto Lagrangian linearises completely giving the term

$$L = (1/2)((\dot{X}_\mu)^2 - ({X'}_\mu)^2)$$

This simplifies enormously the quantization proceedure.(For more details see [9]).

In order to pass to the second quantized theory , just by imitating the point particle case , we define the string field functional Φ as:

$$\Phi(X) = \langle X | \Phi \rangle$$

which is a multilocal functional defined at all points along the string.To give a clearer picture , let us use a specific representation of the $|X\rangle$ eigenstates , in terms of harmonic oscillators.Then we can think of the string as an infinite number of harmonic oscillators.Hence the Hilbert space of the string states will be a tensor product of an infinite number of Hilbert spaces , each one of them representing the states of one harmonic oscillator.Hence , by generalising the well-known formula of the second quantized wave function for the case of point particles , (one harmonic oscillator)

$$|\phi\rangle = \sum_{n=0}^{\infty} \phi_n |n\rangle$$

where $|n\rangle$ are the eigenstates of the Hamiltonian , we get for our string functional the following expression:

$$|\Phi\rangle = \sum_{(n)} \Phi_{(n)} |(n)\rangle$$

where $|(n)\rangle$ is an orthonormal basis for the string Hilbert space.

In order to derive the second quantized Lagrangian for the free open bosonic string , we simply imitate again what we do in the point particle case , i.e. we consider the matrix element between two string states which are infinitesimally close

$$\langle X_1 | e^{-iHd\tau} | X_2 \rangle$$

where

$$H = (\pi/2) \int_0^{\pi} d\sigma (P_i^2 + (X_i^{/})^2/\pi^2)$$

(P_i is the conjugate momentum of X_i) and we just insert a complete set of second quantized field functionals into our action:

$$1 = |\Phi\rangle \int D^2\Phi e^{-\langle\Phi|\Phi\rangle} \langle\Phi|$$

Thus we obtain the second quantized Lagrangian

$$L = \Phi^*(i\partial_\tau - H)\Phi$$

which looks like the second quantized Lagrangian for the Schrödinger equation (non-relativistic free boson).

We now pass to the interactions.Since string field theory is a multilocal theory, in order not to violate locality, causality and momentum conservation, (we remind that only in the LCG where we still work the length of the string is related to the momentum) we postulate that:

"The only interacting string configurations that are allowed in the action are those that instantaneously change the local topology of the strings".(see [9] p.259-260).

From the above , we come to the conclusion that there can exist only five interactions in the LCG .Open and closed strings can only brake, fission and pinch.Thus if we represent an open string field by Φ and a closed string field by Ψ we have that the interacting part of the Lagrangian is

$$L_I = \Phi^3 + \Phi^4 + \Psi^3 + \Phi^2\Psi + \Phi\Psi$$

Let us consider for simplicity the first term Φ^3 , representing an open string that splits into two other open strings.The unique form of the vertex function consistent with momentum conservation and locality is the following:

$$S_3 = \int dp^{+r}\delta(\sum_{r=1}^{3} p^{+r})\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V_{123}\rangle$$

where we have introduced the vertex function

$$|V_{123}\rangle = \int DX_{123}|X_1\rangle|X_2\rangle|X_3\rangle\delta_{123}$$

and

$$DX_{123} = DX_1 DX_2 DX_3$$

(For more details see [9].)

One can then prove that it reproduces the usual Veneziano model (i.e. that it gives the same results as the first quantized theory.) We also mention that the four vertex function has also been written down explicitly (see [9]).As about the closed string fields , we simply mention that they emerge as bounded states of the open string fields.(For more details see [9]).

5.2.2 BRST String Field Theory

The great advantage of the LCG string field theory , appart from being very similar to the point particle case and hence a rather straightforward generalisation is that it is unitary , manifestly ghost-free and it can reproduce the Veneziano model.

However , the LCG field theory is still a gauge-fixed theory.We would like a covariant description.This can be done by using the BRST formalism.The power of this approach is that it can reformulate string field theory in a fully covariant way with the introduction of the Faddeev-Popov ghosts.(Thus all the gauges of the string are in operation and they are not made redundant from the very beginning as it happens in the LCG.) It must be stressed though that even BRST theory is still a gauge-fixed theory.Moreover, the attempts to covariantise the model produces not one but two competing covariant BRST string field theories based on entirely different string topologies and the only apparent link between them is that they both successfully reproduce the Veneziano model.We should mention finally that the BRST approach faces additional problems when dealing with closed strings or superstrings (for more details see [9] and the essay by Mansfield P. in [8].)

In order to derive the light cone field theory we started from the first quantized Lagrangian in the LCG.We shall do exactly the same now.(We shall begin with the free string and then proceed to the interactions). Our starting point is then the Lagrangian of the first quantized theory for the free open bosonic string , i.e. the Nambu-Goto Lagrangian.We choose the conformal gauge now and we obtain a BRST-invariant action by evaluating the F-P determinant and exponentiating it into the action.In order to derive the second quantized Lagrangian,we just consider time-slices as usual and insert a complete set of states

$$1 = |\theta\rangle|X\rangle \int DXD\theta\langle\theta|\langle X|$$

in the expression

$$\langle X_1|e^{-iH\tau}|X_2\rangle$$

where

$$\Phi(X,\theta) = \langle\theta|\langle X|\Phi\rangle$$

Clearly the basic field functional is now a function of the F-P ghost field θ and of the string variable $X(\sigma,\tau)$.We then finally obtain the second quantized Lagrangian

$$L = \langle\Phi|[L_0 - 1]|\Phi\rangle(*)$$

where L_0 is the zeroth Virasoro generator which is now a function of both X and θ.This new Lagrangian is also invariant under the BRST transformation

$$\delta|\Phi\rangle = Q|\theta\rangle$$

where Q is the BRST charge.Because Q is nilpotent , the above Lagrangian can be equivalently written as:

$$L = \langle\Psi|Q|\Psi\rangle$$

The equations of motion corresponding to this Lagrangian is just the reality condition

$$Q|\Psi\rangle = 0$$

One can then further prove that the above action can be gauge-fixed to give both (*) above as well as the LCG action.

We shall now make a brief comment about the interactions.The interacting BRST formalism of Witten does not need a four vertex function but one has to slightly generalise the δ function of the three vertex function of the LCG theory in order to get a new string configuration.The length of all strings is set equal,the parametrisation length is not related to the momentum and the string-midpoint plays a rather special role.(For more details see [9]).

The whole BRST formalism (containing both free and interacting open strings) can be presented by an elegant axiomatic way which is due to Witten.He postulates a non-commutative differential graded algebra $\mathcal{A}$ of which the 1-forms are the string fields.The wedge product is denoted by *.Commutativity is taken in the graded sence as in Connes' quantized forms , so that we have in general

$$w_p * w_q \neq (-1)^{pq} w_q * w_p$$

for a p-form w_p and a q-form w_q.The exterior derivative is the BRST charge Q.There is a linear functional on $\mathcal{A}$ denoted by $\int$ which is in fact a combination of integration and trace.With these ingredients string field theory is built up from an action which is the integral of a Chern-Simons form:

$$S = \int A * QA + (2/3) A * A * A$$

The action is invariant under the gauge transformation

$$\delta A = Q\Lambda + A * \Lambda + \Lambda * A$$

where Λ is an arbitrary 0-form(gauge parameter).To make contact with the Veneziano model,the integrals are actually evaluated as path integrals on the world-sheet.Elements of this string algebra behave much like infinite-matrix-valued differential forms.

To be more concrete,if the parametrisation midpoint of the string is singled out as a special point, then it is possible to define a closed algebra among string field functionals.A string represented by $|A\rangle$ can be joined with a string $|B\rangle$ such that their midpoints exactly coincide.By contracting over their oscillator states,we are left with yet another string of equal length,so we have defined a process similar to multiplication.For example the abstract notation

$$A * B = C$$

means concretely

$$\langle A_1 | \langle B_2 | V_{123} \rangle = |C_3\rangle$$

where we have contracted over the first and second harmonic oscillators and are left with a string defined over the third harmonic oscillators.Because the strings all have equal parametrisation lengths, the product rule closes if the parametrisation midpoint is singled out as a special point;i.e. the product of two strings of length one equals another string of length one.We can define gauge transformations as described above and the curvature as

$$F = QA + A * A$$

Then we find that

$$\delta F = F * \Lambda - \Lambda * F$$

If we define the integration as an operation preserving

$$\int A * B = (-1)^{AB} \int B * A$$

then we can find two invariants,a surface term and the action itself,namely

$\int F * F$=surface term

and

$$L = A * QA + (2/3) A * A * A$$

which is a Chern-Simons form.

The remarkable nature of this approach is that the whole theory now can be formulated via a set of five axioms:

(1).Existence of a nilpotent derivation, $Q^2 = 0$

(2).Associativity of the * product, $(A * B) * C = A * (B * C)$

(3).Leibnitz rule, $Q(A * B) = QA * B + (-1)^A A * QB$

(4).The product rule, $\int A * B = (-1)^{AB} \int B * A$

(5).The integration rule, $\int QA = 0$

where

$(-1)^A$ is -1 if A is odd and +1 if A is even.

We define the integration operation as:

$$\int \Phi = \int DX \prod_{0 \leq \sigma \leq \pi/2} \delta(X(\sigma) - X(\pi - \sigma)) \Phi(X)$$

The disadvantages of the BRST approach were mentioned at the beginning of this subsection.We would like to comment on the existince of two BRST string field theories.Although this is not so obvious, the BRST vertex $|V_{123}\rangle$ when the external particles are placed on-shell,is equivalent to any other on-shell vertex function that can be reached by a conformal transformation generated by L_n,which in turn is equivalent to the Caneschi-Schwimmer-Veneziano vertex found in the early days of the dual model.(CSV vertex).Thus the three string vertex function found by Witten for the BRST formalism yields exactly the same on-shell matrix elements as the old CSV vertex,even though the geometry of the BRST vertex is completely different.

Moreover,as we mentioned previously,for the closed string and superstring the BRST formalism breaks down because the "ghost counting" comes out all wrong.(For more details about these "defects",see for instance [9].)

Let us make a final comment,concerning the origin of these five axioms.It is true that the origin of these axioms is obscure.These five axioms, in accordance to the five axioms of homology (see for instance Cartan-Eilenberg "Homological algebra"), can in principle apply equally well to any homology.We have no understanding therefore of where these axioms came from.What we would like to have in addition is a set of physical principles,out of which the whole theory would emerge.For example,general relativity comes from the two principles of general covariance and the equivalence principle.These principles in turn can be formulated into the five axioms of homology.The same holds for gauge theories.The question therefore is:

"Are there any underlying geometric or physical principles that will allow us to derive these five axioms from them?"

Chapter 6

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