Theology and the Fundamental Sciences I: Theology and Mathematical Logic

(†*Dedicated to the memory of Archbishop Christodoulos of Athens and all Greece.*)

Suppose that one asks the following intriguing question:

Does every problem have a solution?

What would you answer? Most people think that every problem does have a solution (at least one, some may have more than one). But is this true? The answer is *"NO"*! It seems that *questions are more than answers!*

Kurt Gödel (1906-1978) was an Austrian mathematician (his birth place today belongs to the Czech Rebublic, during the beginning of the 20th century it was in Austria-Hungary). He is considered to be the most significant logician of the 20th century and one of the most important logicians of all time. His main contribution in mathematical logic is a pair of theorems known as the famous *incompleteness theorems* which he published in 1931 when he was 25 years old, one year after completing his Ph.D thesis at the University of Vienna. Gödel started his career studying theoretical physics but he soon turned to mathematical logic. He was mainly influenced by the works of I. Kant and B. Russell. He was also a close friend of A. Einstein when they were both at the Institute for Advanced Study in Princeton. Unfortunately he had a sad end: although he had always been rather eccentric, (for example he used to wear a long coat even in the summer), during the last couple of years of his life he suffered from paranoia and amongst his delusions was the belief that unknown villains were trying to kill him by poisoning his food; he refused to eat any food at all and thus died of "malnutrition and inanition caused by personality disturbance" in Princeton Hospital.

Avoiding technical terms and using simple words, the incompleteness theorems say the following:

For any computable axiomatic system which is at least rich enough in order to be able to describe the basic arithmetic of the natural numbers (this is the arithmetic of natural numbers that we learn in primary school, which includes the 4 basic operations: addition, subtraction, multiplication and division, this is called the *Peano arithmetic*), *then:*

1. *If the system is consistent, then it cannot be complete* (this is called the *first incompleteness theorem*).

2. *The consistency of the axioms cannot be proved within the system.*

In other words the (first) theorem says that if a system is consistent, then it cannot be complete, which means that there will always be statements (at least 1), which are called π_1-*sentences*, for which we cannot decide whether they are true or not. The reason why it is called the incompleteness theorem is the following: if we take a specific logical system which is consistent, then according to the above theorem there is at least 1 sentence for which we cannot prove if it is true or not. Suppose there is only one such sentence and since we cannot prove if this sentence is true or not, we consider as an additional axiom of the initial logical system that this sentence is true. Hence, we have a new extended system which has one additional axiom. But then the theorem can be applied again and it implies that for this new system there is at least one sentence which cannot be proved if it is true or not, this process can be continued for ever which means that logic is incomplete. This theorem in fact says more than the above: We may only consider logical systems which are complete at the expense of being inconsistent. Thus *there are no logical systems which are both consistent and complete.*

In science (but also in our relations with other humans including politics), we mainly insist on consistency. This means that in a theory a statement must not be both true and false. *If this happens, then the theory is unable to make reliable predictions*. Yet if we insist on consistency, the incompleteness theorem says that there will be statements (at least one), for which we shall not be able to decide whether they are true or false, namely the system will be incomplete. If on the contrary we want a complete system, namely a system in which every question has an answer, that, inescapably, will contain inconsistencies. Make your choice!

The proof of even the first incompleteness theorem is long and complicated: the author recalls that as a graduate student in Oxford, he attended a whole course which only contained the proof of this theorem! Just for completeness, we shall only mention that the proof is based on what consequently has been called *Gödel's numbering argument*. It immitates *Georg Cantor's* famous *diagonal argument* which proves that the real numbers are more than the integer numbers (they are both infinite but the former are *uncountable* whereas the later are *countable*).

There are many examples of π_1-sentences: perhaps the simplest one is the *parallel postulate* in Euclidean geometry, a statement which can neither be proved nor disproved from Euclid's 4 remaining well-known axioms of incidence, order, congruence and continuity; in other words this sentence is *undecidable* from the other 4 axioms.

Two important examples of π_1-sentences come from mathematical logic itself: as P. J. Cohen proved in 1960's, (in fact Cohen's results are widely considered as the second most significant contributions in mathematical logic during the 20th century, after the incompleteness theorems), both the *continuoum hypethesis* and the *axiom of choice* can neither be proved nor disproved from the Zermelo-Fraenkel Axioms of Set Theory; in other words both these sentences are undecidable from the ZF Axioms.

The incompleteness theorems have profound consequences not only in mathematics and natural sciences (like physics, in particular *M-Theory* which is the modern extension of superstring theory), but also in philosophy and in fact in many other sciences like computer science, cognitive science, physiology, psychology etc. These consequences, until the present day, they have not been fully appreciated and studied, as pointed out by J.R. Lucas, R. Penrose and many other contemporary philosophers and scientists. Without being far from the truth one could claim that the incompleteness theorems constitute a more rigorous and concise version of many of the statements contained in L. Wittgenstein's *"Tractatus"*, a cornerstone of 20th century philosophy. Moreover the incompleteness theorems are closely related to the *theory of deconstruction* in literary theory and philosophy (by Jacques Derrida). More generally, in the light of J.-F. Lyotard's famous definition of the postmodern condition as "incredulity towards meta-narratives", we can mention V. Tasic's remark that mathematical truth belongs to the "the most stuborn meta-narratives of Western culture". Concerning physics, even theoretical physicists like Steven Winberg (Nobel Laureate in Physics 1979, one of the fathers of the so-called *standard model*), seem to realise that a physical *theory of everything*, if it exists at all, (like string/M-Theory or twistor theory), *it cannot be mathematically consistent* due to the Incompleteness Theorems. Moreover, when the incompleteness theorems appeared, they completely destroyed D. Hilbert's programme which conjectured that there must exist a set of axioms sufficient for all mathematics.

Gödel's incompleteness theorems are closely related to the work of A. Turing and computer science [in particular to the *Entscheidungsproblem (decision problem)]*, as well as to many mathematical problems (like for instance the *word problem for finitely presented groups*, the *Whitehead problem in K-theory* or the *classification problem for 4-manifolds and the existence of exotic structures due to S.K. Donaldson*). All the above are precisely further examples of π_1-sentences. Moreover the incompleteness theorems are related to the so-called *liar paradox*:
"The sentence below is false"
"The sentence above is true",
and the *Russell paradox* (one amusing version of which is the *barber's paradox*: "the barber saves only those who cannot save themselves, who saves the barber?") Not surprisingly, some of these paradoxes have their origins in ancient greek literature and philosophy: in 6th century BC, the philosopher and poet Epimenides, himself a Cretan, wrote: *"The Cretans are always liars"* and the

philosopher Eubulides of Miletus in the 4th century BC wrote: *"A man says that he is lying. Is what he says true or false?"* Let us not forget the famous phrase attributed to the philosopher Socrates: *"I know one thing, that I know nothing"*. Hence questions like: "Can the all-mighty God create a stone so large that he cannot lift?" demonstrate the flaws of man-made logic (rather than...incompetence of God). Let us also remember the following phrase of the Letter of St Paul to the Collosians (Chapter 2, verse 8): "Beware lest any man spoil you through philosophy and vain deceit, after the tradition of men, after the rudiments of the world, and not after Christ." (Not to be misunderstood: the Apostle is not against philosophy, he merely emphasises its limitations).

Gödel himself was religious (a catholic christian). In 1970 he produced an ontological argument for the existence of God, which was a refinement of an original ontological argument due to St Anselm of Canterbury (1033-1109). Gödel never published his argument, although he repeatedly showed it to various friends. It was first published in 1987, nine years after his death.

St Anselm's original argument is as follows (Proslogion II):

"...Hence, even the fool is convinced that something exists in the understanding, at least, than which nothing greater can be conceived. For, when he hears of this, he understands it. And whatever is understood, exists in the understanding. And assuredly that, than which nothing greater can be conceived, cannot exist in the understanding alone. For, suppose it exists in the understanding alone: then it can be conceived to exist in reality; which is greater. Therefore, if that, than which nothing greater can be conceived, exists in the understanding alone, the very being, than which nothing greater can be conceived, is one, than which a greater can be conceived. But obviously this is impossible. Hence, there is doubt that there exists a being, than which nothing greater can be conceived, and it exists both in the understanding and in reality..."

One can rewrite St Anselm's original argument in a more modern language using *reductio ad absurdum*. The following steps more closely follow St Anselm's line of reasoning:
1. God is the entity greater than which no entity can be conceived.
2. The concept of God exists in human understanding.
3. God does not exist in reality but only in understanding (assumed in order to refute).
4. The concept of God existing in reality exists in human understanding.
5. If an entity exists in reality and in human understanding, this entity is greater than it would have been if it existed only in human understanding (a statement of existence as a perfection).
6. From $1, 2, 3, 4$, and 5 an entity can be conceived that is greater than God, the

entity greater than which no thing can be conceived (logical self-contradiction).
7. Assumption 3 is wrong, therefore, God exists in reality (assuming $1, 2, 4$, and 5 are accepted as true).

The step which has received most criticism from philosophers and logicians throughout the ages is Step 5. The use of such arguments for the christian belief is arguable: It has been reported that the only person which has been converted to the christian belief by an ontological argument was the famous Oxford professor and writer C.S Lewis (1898-1963). Many ontological arguments for the existence of God appeared subsequently, the most famous are those of R. Descartes and G. Leibniz. Also *parodies* of the ontological arguments appeared.

Gödel's refined argument is as follows (this requires some familiarity with modern modal logic):

Definition 1: x is God-like iff x has as essential properties those and only those properties which are positive.
Definition 2: A is an essence of x iff for every property B, x has B necessarily iff A entails B.
Definition 3: x necessarily exists iff every essence of x is necessarily exemplified.
Axiom 1: If a property is positive, then its negation is not positive.
Axiom 2: Any property entailed by - i.e., strictly implied by - a positive property is positive.
Axiom 3: The property of being God-like is positive.
Axiom 4: If a property is positive, then it is necessarily positive.
Axiom 5: Necessary existence is positive.
Axiom 6: For any property P, if P is positive, then being necessarily P is positive.
Theorem 1: If a property is positive, then it is consistent, i.e., possibly exemplified.
Corollary 1: The property of being God-like is consistent.
Theorem 2: If something is God-like, then the property of being God-like is an essence of that thing.
Theorem 3: Necessarily, the property of being God-like is exemplified.

For the experts really, using symbolic logic, the above argument can be written as follows:

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\begin{array}{rl}
\text{Ax. 1.} & \bullet\, \forall x\{[\varphi(x) \to \psi(x)] \wedge P(\varphi)] \to P(\psi)\} \\
\text{Ax. 2.} & P(\neg\varphi) \iff \neg P(\varphi) \\
\text{Th. 1.} & P(\varphi) \to \Diamond\, \exists x\, [\varphi(x)] \\
\text{Df. 1.} & G(x) \leftrightarrow \forall\varphi[P(\varphi) \to \varphi(x)] \\
\text{Ax. 3.} & P(G) \\
\text{Th. 2.} & \Diamond\, \exists x\, G(x) \\
\text{Df. 2.} & \varphi\ ess\ x \iff \varphi(x) \wedge \forall\psi\{\psi(x) \to \bullet\, \forall x[\varphi(x) \to \psi(x)]\} \\
\text{Ax. 4.} & P(\varphi) \to \bullet\, P(\varphi) \\
\text{Th. 3.} & G(x) \to G\ ess\ x \\
\text{Df. 3.} & E(x) \leftrightarrow \forall\varphi[\varphi\ ess\ x \to \bullet\, \exists x\, \varphi(x)] \\
\text{Ax. 5.} & P(E) \\
\text{Th. 4.} & \bullet\, \exists x\, G(x)
\end{array}
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More formal apects and more details can be found in the following books:

• *Kurt Gödel, Collected Works:* Volume I: *Publications 1929-1936*, Volume II: *Publications 1938-1974*, Volume III: *Unpublished Essays and Lectures*, Volume IV: *Correspondence*, *Oxford University Press*, 1994.

For an introduction to modern modal logic the interested reader can start from the following very good book:

• *Alvin Plantinga: "The Nature of Necessity"*, Clarendon Library of Logic and Philosophy, Oxford University Press, 1979.

For a review on the relation between mathematics and modern philosophy (in particular postmodernism and the work of J.-F. Lyotard), the interested reader may have a look at the following book:

• *Vladimir Tasic: "Mathematics and the Tools of Postmodern Thought"*, Oxford University Press, 2001.

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